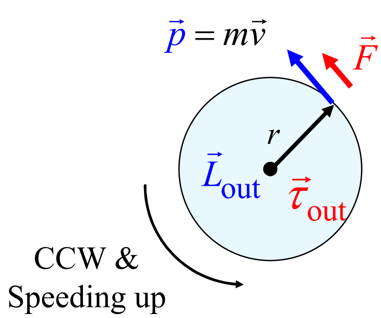
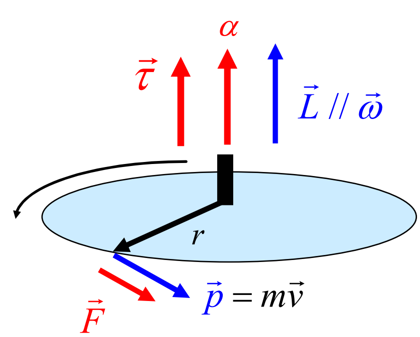
Topic 5B: Rotational Energy & Angular Momentum

Linear Variables	Rotational Variables
$\vec{p} = m\vec{v}$	$\vec{L} = \vec{r} \times \vec{p}$ ($\vec{L} = I\vec{\omega}$ for axis with symmetry)
$\vec{F}_{\text{net}} = m\vec{a} = \dfrac{d\vec{p}}{dt}$	$\vec{\tau}_{\text{net}} = \vec{r} \times \vec{F} = I\vec{\alpha} = \dfrac{d\vec{L}}{dt}$
$K_{\text{trans}} = \frac{1}{2}mv^2 = \dfrac{p^2}{2m}$	$K_{\text{rot}} = \frac{1}{2}I\omega^2 = \dfrac{L^2}{2I}$

$$K_{\text{tot}} = K_{\text{trans, CM}} + K_{\text{rot}} = \tfrac{1}{2}mv^2 + \tfrac{1}{2}I\omega^2$$

Angular Momentum & Torque





$\vec{L} = \vec{r} \times \vec{p}$ where $\vec{L} = I\vec{\omega}$ for rotation about symmetry axis

$$\vec{\tau} = \vec{r} \times \vec{F} = \frac{d\vec{L}}{dt}$$

Angular Momentum & Torque Derivation

$$\boxed{\frac{d\vec{L}}{dt}} = \frac{d}{dt}(\vec{r}\times\vec{p}) = \left(\frac{d\vec{r}}{dt}\times\vec{p}\right) + \left(\vec{r}\times\frac{d\vec{p}}{dt}\right)$$

$$= (\vec{v}\times m\vec{v}) + \left(\vec{r}\times\frac{d\vec{p}}{dt}\right)$$

$$= \left(\vec{r}\times\frac{d\vec{p}}{dt}\right) \text{ where } \vec{v}\times\vec{v} = 0$$

$$\boxed{\frac{d\vec{L}}{dt} = \vec{r}\times\vec{F} = \vec{\tau}}.$$

If $\boxed{\vec{\tau}_{\text{net}} = 0}$, then $\vec{L}$ is constant.

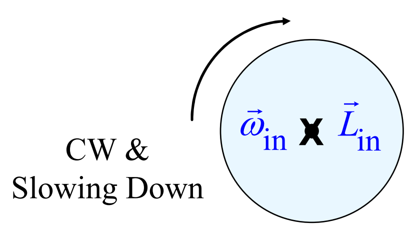
$\vec{L}_{\text{i}} = \vec{L}_{\text{f}}$ where $\vec{L} = I_{\text{i}}\vec{\omega}_{\text{i}}$

$$\boxed{I_{\text{i}}\vec{\omega}_{\text{i}} = I_{\text{f}}\vec{\omega}_{\text{f}}}$$

Question #6.1: Torque

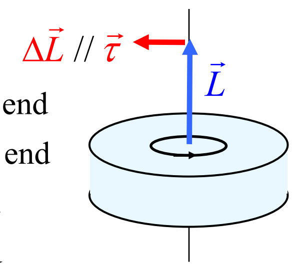
A disk rotates clockwise and is slowing down as shown. In what direction does the net torque point?

(a) Up
(b) Right
(c) Out of page
(d) Into page



Question #6.2: Angular Momentum & Torque

The angular momentum vector L for a spinning wheel points upward along the axis. To make L and the top end of the axis move left, you must exert forces on the top & bottom ends of the axle in which directions?



(a) Push left on top end & push right on bottom end

(b) Push right on top end & push left on bottom end

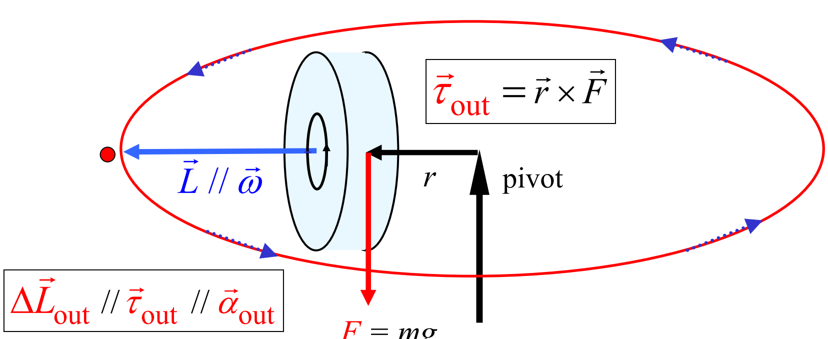
(c) Pull out on top end & push in on bottom end

(d) Push in on top end & pull out on bottom end

Angular Momentum & Torque: COOL Bicycle Wheel

If a gyroscope is mounted on a pivot as shown, the gyroscope does not fall down! Instead it "turns" in a circle - known as precession.

The change in the direction of the angular momentum vector is in the direction of the net torque acting on it. (Bicycle wheel demonstration)



Wheel would rotate counterclockwise (front view) IF it were not spinning.

Total Energy: Translational and Rotational

$$\boxed{K_{\text{tot}} = K_{\text{trans, CM}} + K_{\text{rot}}}$$

$$K_{\text{tot}} = \tfrac{1}{2} m v^2 + \tfrac{1}{2} I \omega^2$$

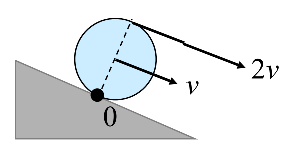
where $\omega = \dfrac{v}{R}$; $\boxed{I = \beta m R^2}$ $\beta = \tfrac{1}{2}$ for disk, etc.

$$K_{\text{tot}} = \tfrac{1}{2} m v^2 + \tfrac{1}{2}\left(\beta m R^2\right)\left(\frac{v^2}{R^2}\right)$$

$$\boxed{K_{\text{tot}} = \tfrac{1}{2} m v^2 \left(1 + \beta\right)}$$

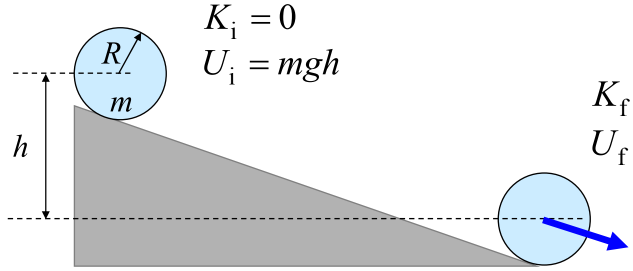
Convert U_G to K: Rolling w/o Slipping DOWN Incline

If an object with moment of inertia I starts rolling down an incline and drops by height h, find its final velocity. Assume that the object does not slip, so that its velocity $v = \omega R$.



Initial Position

$K_i = 0$
$U_i = mgh$



$K_f = \tfrac{1}{2} m v^2 (1 + \beta)$
$U_f = 0$

Final Position

Convert U_G to K: Rolling w/o Slipping DOWN Incline

$$\boxed{\Delta K + \Delta U + f\,\Delta s = W_{\text{ext}}} \quad \text{where} \quad W_{\text{ext}} = f = 0$$

$$\left(K_{\text{f}} - K_{\text{i}}\right) + \left(U_{\text{Gf}} - U_{\text{Gi}}\right) = 0$$

$$\left(\tfrac{1}{2} m v^2 \left(1+\beta\right) - 0\right) + \left(0 - mgh\right) = 0$$

$$\boxed{\tfrac{1}{2} m v^2 \left(1+\beta\right) = mgh}$$

$$\boxed{v = \sqrt{\frac{2 g h_{\text{cm}}}{1+\beta}}}$$

Question #6.3: Two Rolling Cylinders - different R

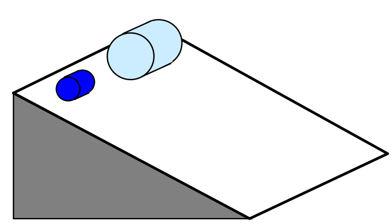
Two cylinders have the same mass but one has twice the radius of the other. If both are placed at the top of a ramp and released, which one reaches the bottom first?

(a) bigger one

(b) smaller one

(c) same

(d) insufficient information



Question #6.4: Two Rolling Cylinders - different m

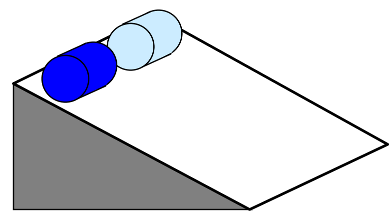
Two cylinders have the same radius but one has twice the mass of the other. If both are placed at the top of a ramp and released, which one reaches the bottom first?

(a) heavier one

(b) lighter one

(c) same

(d) insufficient information



Question #6.5: Cylinder and Ring - same m, R

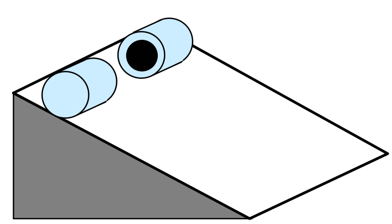
A cylinder and a ring have the same radius and same mass. If both are placed at the top of a ramp and released, which one reaches the bottom first?

(a) cylinder

(b) ring

(c) same

(d) insufficient information



Question #6.6: Cylinder and Ring - unknown m, R

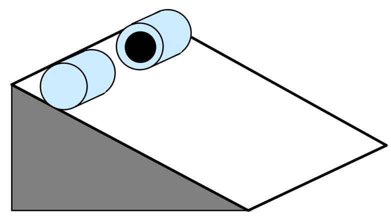
A cylinder and a ring have unknown mass and size. If both are placed at the top of a ramp and released, which one reaches the bottom first?

(a) cylinder

(b) ring

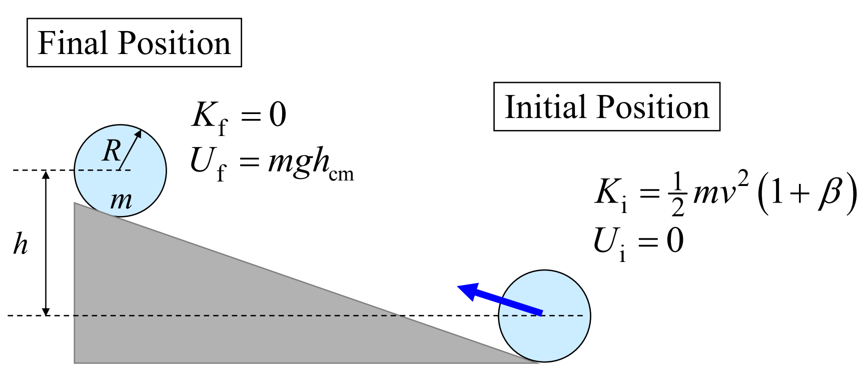
(c) same

(d) insufficient information



Convert K to U_G : Rolling w/o Slipping UP Incline

If an object with moment of inertia $I = \beta v^2$ and velocity v rolls up an incline, find the change in height h where it momentarily stops.



Convert K to U_G : Rolling w/o Slipping UP Incline

$$\boxed{\Delta K + \Delta U + f\,\Delta s = W_{\text{ext}}} \quad \text{where} \quad W_{\text{ext}} = f = 0$$

$$\left(K_{\text{f}} - K_{\text{i}}\right) + \left(U_{\text{Gf}} - U_{\text{Gi}}\right) = 0$$

$$\left(0 - \tfrac{1}{2} m v^2 \left(1+\beta\right)\right) + \left(m g h_{\text{cm}} - 0\right) = 0$$

$$\boxed{\tfrac{1}{2} m v^2 \left(1+\beta\right) = m g h_{\text{cm}}}$$

$$\boxed{h_{\text{cm}} = \frac{\left(1+\beta\right) v^2}{2g} = \frac{K_{\text{tot}}}{mg}}$$

Question #6.7: Cylinder and Ring - same v

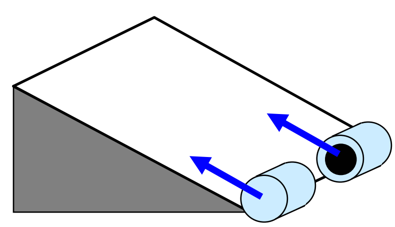
A cylinder and a ring have unknown mass and size, but they have the same velocity v. If both encounter a ramp and travel up it until stopping, which one has a greater change in height?

(a) cylinder

(b) ring

(c) same

(d) insufficient information



Question #6.8: Cylinder and Ring - same K_{tot}

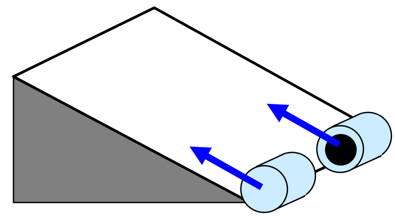
A cylinder and a ring have unknown mass and size, but they have the same total kinetic energy K_{tot}. If both encounter a ramp and travel up it until stopping, which one has a greater change in height?

(a) cylinder

(b) ring

(c) same

(d) insufficient information



Question #6.9: Ball and Box - same m, v

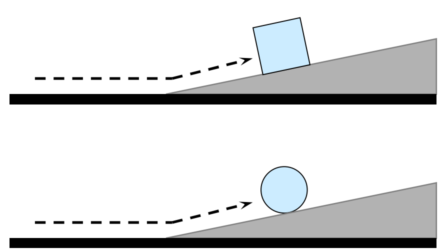
A ball and box have the same mass and are moving with the same velocity across a horizontal floor. The ball rolls without slipping and the box slides without friction. If both encounter a ramp and travel up it until stopping, which one has a greater change in height?

(a) box

(b) ball

(c) same

(d) insufficient information



Question #6.10: Kinetic Energy of Rolling Ball

A bowling ball rolls along the floor without slipping. What is the ratio of its rotational kinetic energy to its translational kinetic energy?

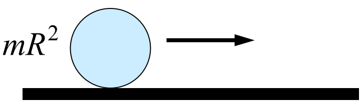
(a) 2

(b) $\frac{1}{5}$

(c) $\frac{2}{5}$

(d) $\frac{1}{2}$

$I = \frac{2}{5} mR^2$



Conservation of Angular Momentum

If $\boxed{\vec{\tau}_{\text{net}} = 0}$, then $\vec{L}$ is constant.

$$\vec{L}_{\text{i}} = \vec{L}_{\text{f}} \quad \text{where} \quad \vec{L} = I_{\text{i}}\vec{\omega}_{\text{i}}$$

$$\boxed{I_{\text{i}}\vec{\omega}_{\text{i}} = I_{\text{f}}\vec{\omega}_{\text{f}}}$$

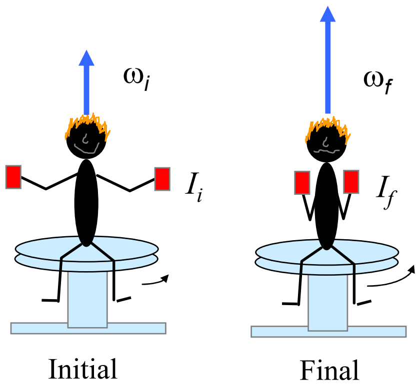
Two Types of Problems:

Changing Moment of Inertia I

"Collisions" of Two Rotating Objects

Prob. #6.1: Changing Moment I ($\tau_{net} = \Delta L = 0$)

A student sits on a rotating stool with her arms extended and a weight in each hand. The total moment of inertia is I_i and she rotates at angular velocity ω_i. She then pulls her arms in and reduces the moment of inertia to I_f. What is her final angular velocity ω_f?



Use $L_i = L_f$ since no external torque acts on the system.

$$I_i \omega_i = I_f \omega_f$$

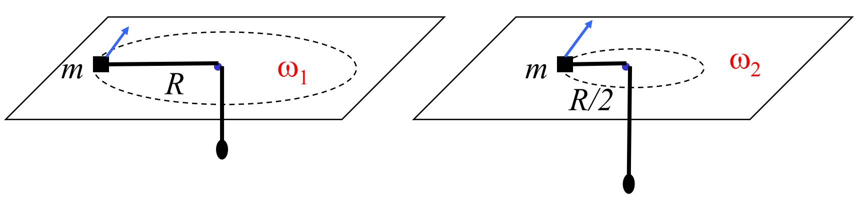
$$\omega_f = \left(\frac{I_i}{I_f}\right) \omega_i$$

Adapted from Physics 111 course at UIUC.

Question #6.11: Changing Moment I ($\tau_{net} = \Delta L = 0$)

A puck is attached to a string and slides on a frictionless table in a circle of radius R with angular velocity ω_0. The string passes through a hole in the table at the center of the circle. If you pull on the string and decrease the radius to $R/2$, what is the new angular velocity ω_f?

(a) ω_0 (b) $2\omega_0$ (c) $4\omega_0$ (d) $8\omega_0$



Question #6.12: Changing Moment I ($\tau_{net} = \Delta L = 0$)

A student sits on a stool rotating at ω_0. She pulls her arms in and halves her moment I. This doubles her angular velocity (due to L conservation). What happens to her kinetic energy?

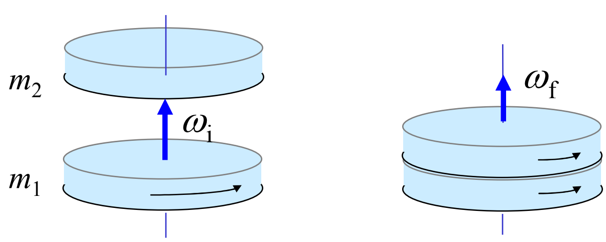
(a) Increases

(b) Decreases

(c) Stays the same

Prob. #6.2: Two Disk "Collision" ($\tau_{net} = \Delta L = 0$)

Disk #1 (mass m_1, radius R) rotates with initial ω_i. Disk #2 (mass m_2, radius R) is not rotating and is dropped on top of disk #1. If there is friction between the disks and they rotate together, find the final ω_f.



$$L_i = I_i\omega_i = \tfrac{1}{2}m_1R^2\omega_i \qquad L_f = \left(\tfrac{1}{2}m_1R^2 + \tfrac{1}{2}m_2R^2\right)\omega_f$$

Use $L_i = L_f$ since no external torque acts on the system.

$$\tfrac{1}{2}m_1R^2\omega_i = \left(\tfrac{1}{2}m_1R^2 + \tfrac{1}{2}m_2R^2\right)\omega_f \quad \Rightarrow \quad \omega_f = \left(\frac{m_1}{m_1 + m_2}\right)\omega_i$$

Question #6.13: Two Disk "Collision" ($\tau_{net} = \Delta L = 0$)

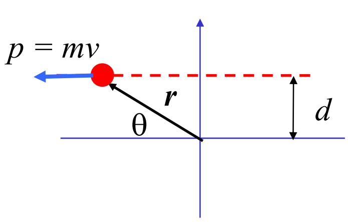
Disk #1 (moment I_0) rotates with initial ω_0. An identical disk #2 rotates in the opposite direction with identical angular speed. If the disks "collide" and eventually rotate together, what is the final ω_f?

(a) $-0.5\omega_0$ (b) ω_0 (c) $0.5\omega_0$ (d) 0

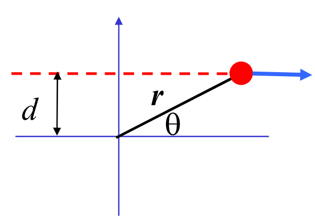
Angular Momentum of "Free" Particle

A particle (mass m, speed v) moves along a horizontal line as shown. Find its angular momentum L with respect to an axis at the origin.

Objects have angular momentum even if they do not "rotate" !



L **OUT** of plane
(using right-hand rule)



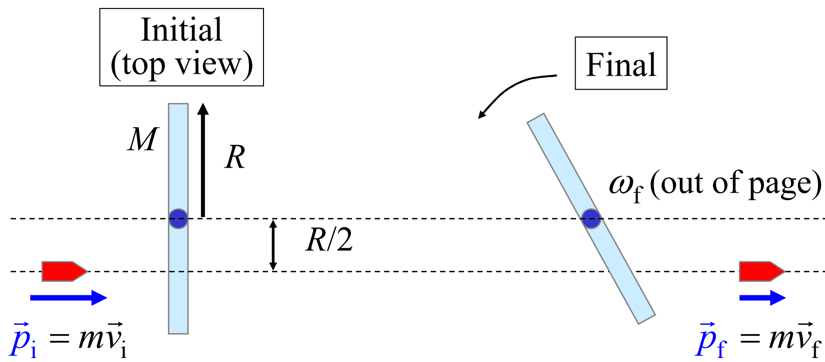
L **INTO** plane

$$\left|\vec{L}\right| = \left|\vec{r} \times \vec{p}\right| = r\left(mv\right)\sin\theta$$

$$\boxed{L = mvd} \quad \text{where} \quad d = r\sin\theta$$

Prob. #6.3a: Bullet & Rod "Collision" ($\tau_{net} = \Delta L = 0$)

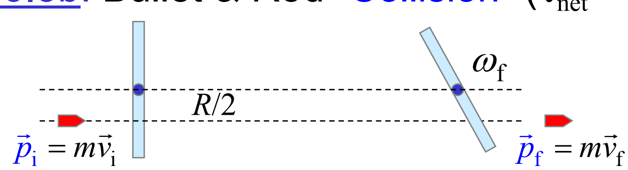
A horizontal rod (mass M) is pivoted at the center. A bullet of mass m is shot through the rod as shown. The initial speed of the bullet is v_i and the final speed is v_f. Find the angular velocity ω_F of the rod after the collision.



Rod length = $2R$

$$\boxed{I_{rod}} = \tfrac{1}{12}M(2R)^2 = \tfrac{1}{3}MR^2$$

Prob. #6.3b: Bullet & Rod "Collision" ($\tau_{net} = \Delta L = 0$)



$$\boxed{L_{\text{bullet, i}}} = \left|\vec{r} \times \vec{p}_i\right| = \left(\tfrac{R}{2}\right)(mv_i)$$
(out)

$$\boxed{L_{\text{bullet, f}}} = \left(\tfrac{R}{2}\right)(mv_f) \quad \text{(out)}$$

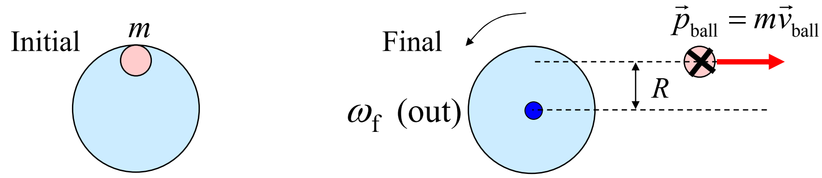
$$\boxed{L_{\text{rod, f}}} = I_{rod}\omega_f = \tfrac{1}{3}MR^2\omega_f$$
(out)

$$\boxed{L_{\text{bullet, i}} + L_{\text{rod, i}} = L_{\text{bullet, f}} + L_{\text{rod, f}}}$$

$$\left(\tfrac{R}{2}\right)(mv_i) + 0 = \left(\tfrac{R}{2}\right)(mv_f) + \tfrac{1}{3}MR^2\omega_f \quad \Rightarrow \quad \boxed{\omega_f = \frac{3m(v_i - v_f)}{2MR}}$$

Prob. #6.4: Ball & Stool "Separation" ($\tau_{net} = \Delta L = 0$)

A student sits on a stool that can rotate (total moment I). She THROWS a ball (mass m, speed v) at a radius R from the axis of rotation. Find the final ω_f of the student+stool (s+s) after she throws the ball.



$$L_{s+s,\,i} = L_{ball,\,i} = 0$$

$$\boxed{L_{s+s,\,f}} = I_{s+s}\omega_f \quad \text{(out)}$$

$$\boxed{L_{ball,\,f}} = \left|\vec{r} \times \vec{p}\right| = R\left(mv_{ball}\right) \quad \text{(in)}$$

$$L_{s+s,\,i} + L_{ball,\,i} = L_{s+s,\,f} + L_{ball,\,f}$$

$$0 + 0 = I_{s+s}\omega_f - R\left(mv_{ball}\right) \Rightarrow \boxed{\omega_f = \frac{Rmv_{ball}}{I_{s+s}}}$$

Question #6.14: Ball & Merry-go-Round ($\tau_{net} = \Delta L = 0$)

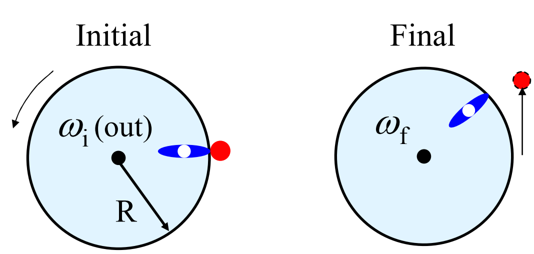
A student rides on the outside of a rotating merry-go-round and holds a ball in her hand. If she RELEASES the ball, the angular MOMENTUM of the ball will:

(a) Increase

(b) Remain the same

(c) Decrease

(d) Insufficient info.



Question #6.15: Ball & Merry-go-Round ($\tau_{net} = \Delta L = 0$)

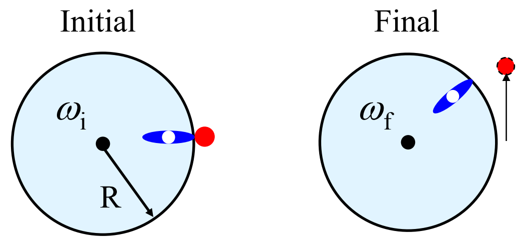
A student rides on the outside of a rotating merry-go-round and holds a ball in her hand. If she RELEASES the ball, the angular VELOCITY of the merry-go-round will:

(a) Increase

(b) Remain the same

(c) Decrease

(d) Insufficient info.



Question #6.16: Ball & Merry-go-Round ($\tau_{net} = \Delta L = 0$)

A student rides on the outside of a rotating merry-go-round and holds a ball in her hand. If she THROWS the ball straight ahead, the angular VELOCITY of the merry-go-round will:

(a) Increase

(b) Remain the same

(c) Decrease

(d) Insufficient info.