Topic #3: Work and Energy

Work $\boxed{W = \int_{s_i}^{s_f} \vec{F} \cdot d\vec{s}}$ uses vector dot product!

Potential Energy $\boxed{\Delta U \text{ (due to conservative F)} = -W \text{ (done by F)}}$

Net Work and Kinetic Energy Theorem (#1)

$$\boxed{W_{\text{net}} = \Delta K}$$

General Work and Energy Theorem (#2)

$$\boxed{W_{\text{ext}} = \Delta K + \Delta U + f\,\Delta s}$$

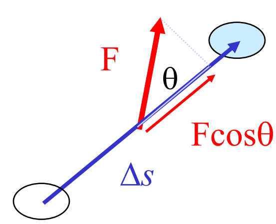
where $K = \frac{1}{2}mv^2;\; U_G = mgh;\; U_S = \frac{1}{2}kx^2;\; f = \mu N$

Definition of Work

The work W done on an object by a force F is a product of the distance Δs traveled by the object and the component of F along the direction of motion (known as dot product between F and Δs).

Constant Force: $\boxed{W = \vec{F} \cdot \Delta\vec{s} = F\,\Delta s \cos\theta}$

Variable Force: $\boxed{W = \int_{s_i}^{s_f} \vec{F} \cdot d\vec{s}}$



Units: Force (kg m/s^2 or N) × Parallel Distance (m)

= Work (kg m^2/s^2 or Joule)

Note: 1 calorie = 4.2 J (food "calories" are actually kilocalories)

What is Work?

- If you push very hard on a wall, do you do work?
- If a string keeps a ball going in a circle, does the string do work?
- If you hold a ball up in the air, do you do work?
- If you push a block up an incline at increasing speed, do you do work?
- If you push a block up an incline at constant speed, do you do work?
- If a ball is dropped, is work done? By what?

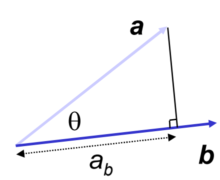
Dot Product using Vector Angle

A dot product between two vectors multiplies the PARALLEL components of the vectors and is a SCALAR.

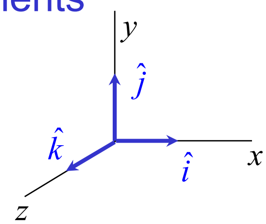
- If two vectors are parallel, then the dot product is the product of their lengths.
- If two vectors are perpendicular, then the dot product is zero.
- If two vectors have an angle θ between them, then the dot product is the product of their lengths times cosθ.

$$\boxed{\vec{A}\cdot\vec{B}} = \left|\vec{A}\right|\left|\vec{B}\right|\cos\theta = \boxed{AB\cos\theta}$$

$$\cos\theta = \frac{\vec{A}\cdot\vec{B}}{AB}$$



Dot Product using Components



$$\hat{i}\cdot\hat{i}=1; \quad \hat{j}\cdot\hat{j}=1; \quad \hat{k}\cdot\hat{k}=1$$

$$\hat{i}\cdot\hat{j}=0; \quad \hat{j}\cdot\hat{k}=0; \quad \hat{i}\cdot\hat{k}=0$$

$$\begin{aligned}\vec{A}\cdot\vec{A} &= \left(A_x\hat{i}+A_y\hat{j}+A_z\hat{k}\right)\cdot\left(A_x\hat{i}+A_y\hat{j}+A_z\hat{k}\right)\\ &= A_x^2\left(\hat{i}\cdot\hat{i}\right)+A_y^2\left(\hat{j}\cdot\hat{j}\right)+A_z^2\left(\hat{k}\cdot\hat{k}\right)+\\ &\quad 2A_xA_y\left(\hat{i}\cdot\hat{j}\right)+2A_yA_z\left(\hat{j}\cdot\hat{k}\right)+2A_xA_z\left(\hat{i}\cdot\hat{k}\right)\end{aligned}$$

$\boxed{A^2 = A_x^2 + A_y^2 + A_z^2}$ (Pythagorean Theorem)

$$\boxed{\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z}$$

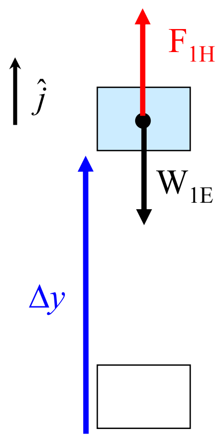
Question #6.1: Dot Product for Two Vectors

If $\vec{A}=\sqrt{2}\,\hat{i}-\sqrt{2}\,\hat{j}$ and $\vec{B}=3\hat{i}+4\hat{j}$ then $\vec{A}\cdot\vec{B}$ equals:

(a) $-\sqrt{2}$ (b) $\sqrt{2}$ (c) $7\sqrt{2}$ (d) $2\sqrt{2}$

Problem #6.1: Net Work for Raised Block

Find the net work done on a block that is raised with a hand force F_{1H} over a vertical distance Δy.



Work by gravity *Work by hand*

$$W_{net} = W_G + W_H$$

$$W_G = \vec{W}_{1E} \cdot \Delta\vec{y} = \left(-mg\ \hat{j}\right) \cdot \left(\Delta y\ \hat{j}\right) = -mg\Delta y$$

$$W_H = \vec{F}_{1H} \cdot \Delta\vec{y} = \left(F_{1H}\ \hat{j}\right) \cdot \left(\Delta y\ \hat{j}\right) = F_{1H}\Delta y$$

$$\boxed{W_{net} = \left(F_{1H} - mg\right)\Delta y}$$

When is the net work zero, positive, or negative?

Question #6.2: Work for Lifting Object

If a block is lifted up, then which statement is true?

(a) You are doing positive work & gravity does negative work.

(b) You are doing positive work & gravity does positive work.

(c) You are doing negative work & gravity does negative work.

(d) You are doing negative work & gravity does positive work.

Question #6.3: Work for Dropped Object

If a block is dropped a distance Δy, then which statement is true?

(a) Work by gravity = mg Δy

(b) Work by gravity = $-$mg Δy

(c) Work by gravity = zero

Question #6.4: Work for Thrown Object

If a ball is thrown at an angle θ from the horizontal and falls a distance Δy, then which statement is true?

(a) Work by gravity = mg Δy

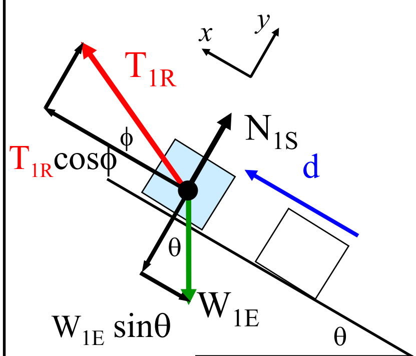
(b) Work by gravity = mg $\cos\theta$ Δy

(c) Work by gravity = mg $\sin\theta$ Δy

(d) Work by gravity = zero

Problem #6.2: Net Work for Pulled Block on Incline

Find the net work done on a block that is pulled by a rope at an angle ϕ up an incline of angle θ over a distance d.



$$W_{net} = W_G + W_T$$

$$W_G = \vec{W}_{1E} \cdot \Delta\vec{x} = (-mg\sin\theta)(d)$$

$$W_T = \vec{T}_{1R} \cdot \Delta\vec{x} = (T_{1R}\cos\phi)(d)$$

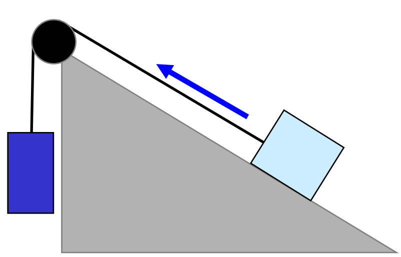
$$\boxed{W_{net} = (T_{1R}\cos\phi - mg\sin\theta)d}$$

If the net work is positive, what happens to the speed of the block?

Question #6.5: Work done by Forces

A box is pulled up a rough ($\mu > 0$) incline by a rope-pulley-weight arrangement as shown. How many forces are doing work on the box?

(a) One
(b) Two
(c) Three
(d) Four



Net Work and Kinetic Energy Theorem (#1)

- If there is some NET work done on an object, then its kinetic energy MUST change!
- For positive net work, the object speeds up.
  For negative net work, the object slows down.
- Why?? Because there is some force parallel to the object's motion that results in $a_{//}$. Remember that $a_{//}$ causes an object to speed up or slow down.

$$\boxed{W_{net}} = \int_{s_1}^{s_2} \vec{F}_{net} \cdot d\vec{s} = \boxed{\Delta K} \quad \text{where} \quad \Delta K = \tfrac{1}{2} m \left(v_f^2 - v_i^2 \right)$$

Net Work and Kinetic Energy Theorem (#1)

Derivation:

$$W = \int_{x_1}^{x_2} F(x)\, dx = \int_{x_1}^{x_2} ma\, dx = m \int_{x_1}^{x_2} \frac{dv}{dt}\, dx$$

$$= m \int_{x_1}^{x_2} \left(\frac{dv}{dx} \right) \left(\frac{dx}{dt} \right) dx = m \int_{x_1}^{x_2} \left(\frac{dv}{dx} \right) (v)\, dx$$

$$\boxed{W} = m \int_{v_1}^{v_2} v dv = \left. \frac{mv^2}{2} \right|_{v_1}^{v_2} = \tfrac{1}{2} mv_2^2 - \tfrac{1}{2} mv_1^2 = \boxed{\Delta K}$$

Question #6.6: Net Work and Motion of Object

If there is net work done on an object, then

(a) its travel direction always changes & it always speeds up/slows down.

(b) its travel direction always changes & it may speed up/slow down.

(c) its travel direction may change & it always speeds up/slows down.

(d) its travel direction may change & it may speed up/slow down.

Question #6.7: Kinetic Energy K

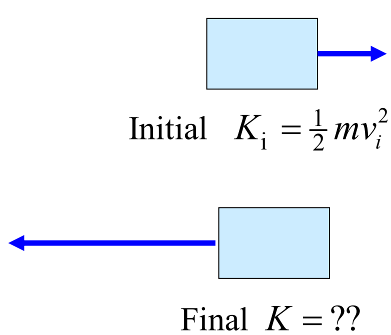
An object has initial kinetic energy K_i. The object then moves in the opposite direction with three times its initial speed. What is the kinetic energy now?

(a) $3\ K_i$

(b) $-3\ K_i$

(c) $9\ K_i$

(d) $-9\ K_i$



Question #6.8: Work and Change in Velocity

A box is pulled by a string with horizontal force F across a horizontal, frictionless table. If the box starts from rest, what is the relationship between the speed $\boldsymbol{v_1}$ after the box has moved a distance d to the speed $\boldsymbol{v_2}$ after the box has moved a distance $2d$?

(a) $v_2 = v_1$

(b) $v_2 = 2v_1$

(c) $v_2 = v_1 / 2$

(d) None of the above.

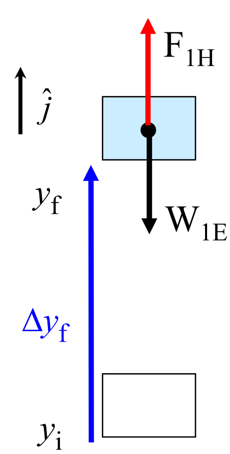
Work and Gravitational Potential Energy U_G

General: $\boxed{\Delta U = -W = -\int_{s_i}^{s_f} \vec{F} \cdot d\vec{s}}$

$\boxed{\text{Gravity}}$: $\vec{W}_{1E} = -mg\,\hat{j}$

$$\Delta U_G = -\int_{y_i}^{y_f} \left(-mg\,\hat{j}\right)\cdot\left(dy\,\hat{j}\right) = mg\int_{y_i}^{y_f} dy$$

$$\boxed{\Delta U_G = mg\left(y_f - y_i\right)}$$



If an object is moved in a "conservative" force field like gravity, U_G depends ONLY on height. The path taken between two heights does not affect the final potential energy.

(This is not the case for non-conservative forces such as friction.)

Question #6.9: Gravitational Potential Energy

If Joe and his twin Jack both reach the top of Half Dome peak in Yosemite, where Joe takes the back route with a gently sloped trail and Jack scales the vertical rock face, which of the following statements is true?

(a) Jack gains more gravitational potential energy than Joe.

(b) Jack gains less gravitational potential energy than Joe.

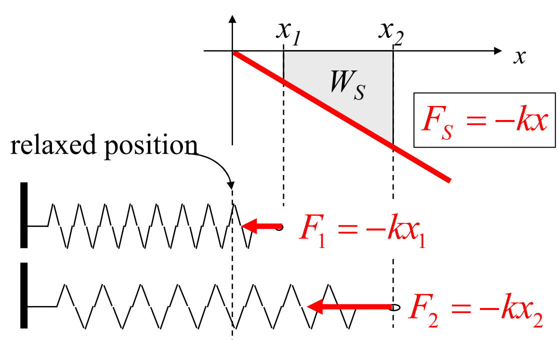
(c) Both gain equal gravitational potential energy.

(d) To compare energies, the height of the peak must be known.

Work and Spring Potential Energy U_S

The force F_S of a spring is proportional to how far Δx it has been expanded or compressed from its relaxed, equilibrium position.

The force is in the opposite direction to its displacement.



$$\Delta U_S = -W_S = -\int_{x_i}^{x_f} F_S \, dx = -\int_{x_i}^{x_f} (-kx)\, dx$$

$$\boxed{\Delta U_S} = \left.\frac{kx^2}{2}\right|_{x_i}^{x_f} = \boxed{\tfrac{1}{2}k\left(x_f^2 - x_i^2\right)}$$

Question #6.10: U_s for Two Stretched Lengths

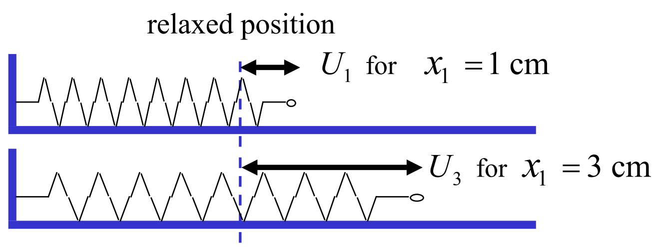
How does U_3 of a spring stretched 3 cm from its natural length compare to U_1 of a spring stretched 1 cm?

(a) $U_3 = 3U_1$

(b) $U_3 = 6U_1$

(c) $U_3 = 9U_1$

(d) $U_3 = 27U_1$



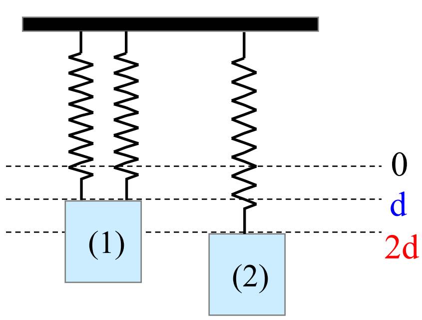
Question #6.11: U_s of Two Spring Setups

Which of the blocks below has the most potential energy stored in its spring(s), relative to the relaxed position at $y = 0$?

(a) Block 1

(b) Block 2

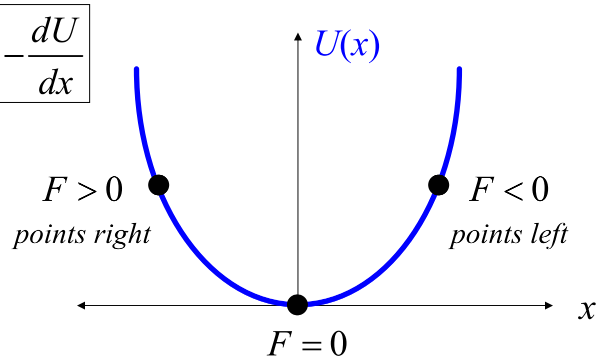
(c) Blocks 1 & 2 same



Potential Energy and Force

Earlier, we saw that potential energy was the integral of the force. Now, we see that the force is therefore the derivative of the potential energy.

$$\boxed{F_x = -\frac{dU}{dx}}$$



If this graph were the potential energy for a mass oscillating on a spring, then where would the spring force be largest?

General Work-Energy Theorem #2

$$\boxed{W_{\text{net}} = \Delta K} \quad \text{(Work-Kinetic Energy Theorem)}$$

$$W_{\text{net}} = W_{\text{ext}} + W_{\text{nc}} + W_{\text{c}}$$

W_{ext} : work done by external forces (hand, rope, etc.)

$W_{\text{nc}} = -f\Delta s$: work done by non-conservative forces (friction f)

$W_{\text{c}} = -\Delta U$: work done by conservative forces (gravity, spring)

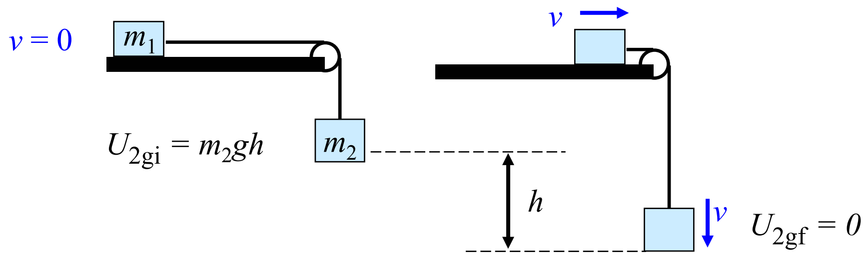
$$W_{\text{ext}} + W_{\text{nc}} + W_{\text{c}} = \Delta K$$

$$W_{\text{ext}} - f\Delta s - \Delta U = \Delta K$$

$$\boxed{W_{\text{ext}} = \Delta K + \Delta U + f\Delta s} \quad \text{(General Work-Energy Theorem)}$$

Problem #6.3: Convert U_G to K

Two blocks of mass m_1 and m_2 are attached as shown. Assume there is no friction and the rope+pulley is massless. Find the speed v of the masses after m_2 has fallen a distance h .



$\boxed{\Delta K + \Delta U + f\,\Delta s = W_{\text{ext}}}$ where $W_{\text{ext}} = f = 0$

$$\left(K_{1f} - K_{1i}\right) + \left(K_{2f} - K_{2i}\right) + \left(U_{1gf} - U_{1gi}\right) + \left(U_{2gf} - U_{2gi}\right) = 0$$

$$\left(\tfrac{1}{2} m_1 v^2 - 0\right) + \left(\tfrac{1}{2} m_2 v^2 - 0\right) + \left(0 - 0\right) + \left(0 - m_2 g h\right) = 0 \Rightarrow \boxed{v = \sqrt{\frac{2 m_2 g h}{m_1 + m_2}}}$$

Question #6.12: Convert U_G to K

Two unequal masses are hung by a massless cord over a pulley in an Atwood machine. After the masses are released from rest, which of the following statements is true?

(a) $\Delta U_G > 0$ and $\Delta K > 0$

(b) $\Delta U_g > 0$ and $\Delta K < 0$

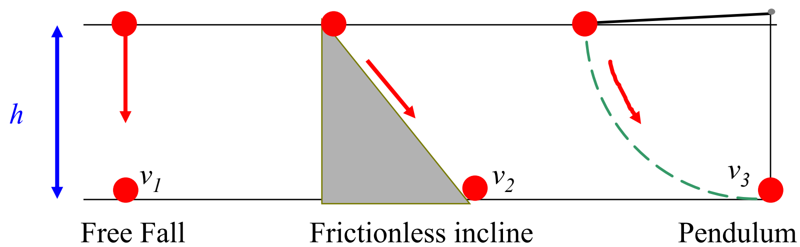
(c) $\Delta U_g < 0$ and $\Delta K > 0$

(d) $\Delta U_g < 0$ and $\Delta K < 0$

Question #6.13: Convert U_G to K

Three objects of mass m begin at height h with zero velocity. One falls straight down, one slides down a frictionless inclined plane, and one swings on the end of a pendulum. What is the relationship between their velocities when they have fallen to the same bottom height?

(a) $v_1 > v_2 > v_3$ (b) $v_1 < v_2 < v_3$ (c) $v_1 = v_2 = v_3$



Do the objects arrive at the ground at the same time?

Question #6.14: Relate U_G and K

Two stones are thrown with the same initial speed at the same instant from the roof of a building. One stone is thrown at an angle of 30° above the horizontal, and the other is thrown horizontally. Neglecting air resistance, whish statement is true?

(a) The stones hit the ground at the same time with equal speeds.

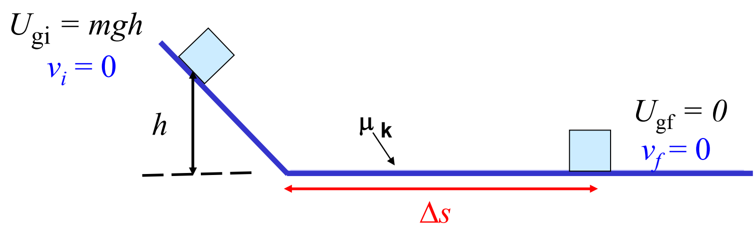
(b) The stones hit the ground at the same time with different speeds.

(c) The stones hit the ground at different times with equal speeds.

(d) The stones hit the ground at different times with different speeds.

Problem #6.4: Convert U_G to Heat

A block slides down a frictionless ramp of height h and then along a rough, horizontal portion of the track that has a kinetic coefficient of friction μ_k. Find the distance Δs that the block travels before stopping.



$$\boxed{\Delta K + \Delta U + f\,\Delta s = W_{ext}} \quad \text{where } W_{ext} = 0$$

$$\left(K_f - K_i\right) + \left(U_{gf} - U_{gi}\right) + f\,\Delta s = 0 \quad \text{where } f = \mu_k mg$$

$$(0-0) + (0 - mgh) + (\mu_k mg)\,\Delta s = 0 \quad \Rightarrow \boxed{\Delta s = \frac{h}{\mu_k}}$$

Question #6.15: Convert K to Heat #2

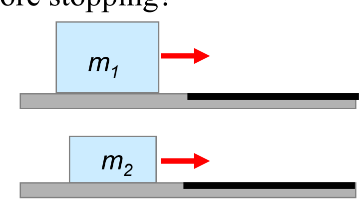
Two blocks have masses m_1 and m_2, where $m_1 > m_2$. They are sliding on a frictionless floor and have the same kinetic energy when they encounter a long rough stretch (i.e. $\mu > 0$) which slows them down to a stop. Which block will go farther before stopping?

(a) m_1

(b) m_2

(c) same distance

(d) insufficient information



Question #6.16: Convert K to Heat #2

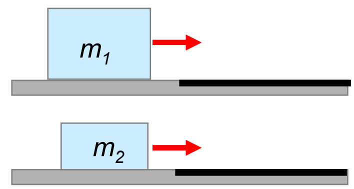
Two blocks have masses m_1 and m_2, where $m_1 > m_2$. They are sliding on a frictionless floor and have the same velocity when they encounter a long rough stretch (i.e. $\mu > 0$) which slows them down to a stop. Which block will go farther before stopping?

(a) m_1

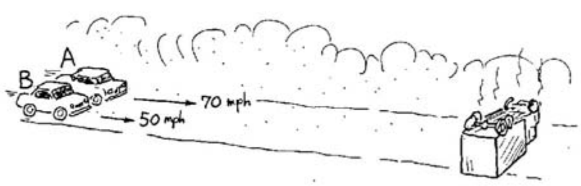
(b) m_2

(c) same distance

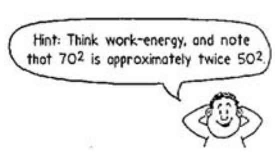
(d) insufficient information



Question #6.17: Convert different K to Heat



On a foggy day, car A moving at 70 mph and another car B moving at 50 mph slam on their brakes when they are neck and neck with each other, in effort to avoid hitting an overturned truck ahead of them. Car B skids to a stop within an inch of the truck and faster car A skids and slams into the truck. From this information the speed of car A hitting the truck is about

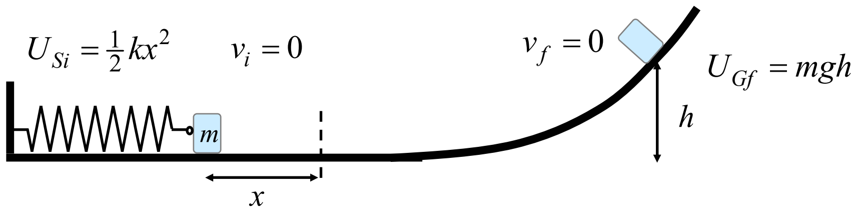


a) 20 mph
b) 30 mph
c) 35 mph
d) 50 mph
e) cannot answer without more information.

Adapted from Figuring Physics - Hewitt

Problem #6.5: Convert U_S to U_G

A block on a frictionless horizontal surface is pushed against a spring that is compressed a distance x. The block is then released and travels up an inclined surface. Find the maximum height h of the block.



$$\Delta K + \Delta U + f\Delta s = W_{\text{ext}} \qquad \text{where } W_{\text{ext}} = f = 0$$

$$\left(K_f - K_i\right) + \left(U_{Gf} - U_{Gi}\right) + \left(U_{Sf} - U_{Si}\right) = 0$$

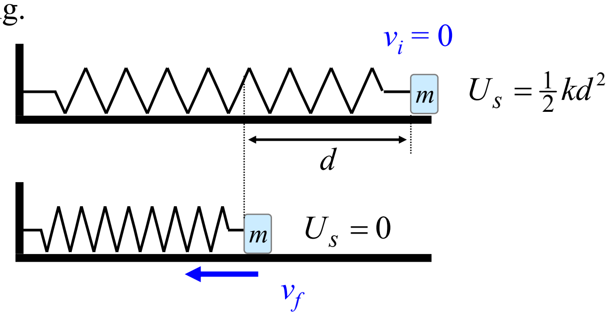
$$\left(0 - 0\right) + \left(mgh - 0\right) + \left(0 - \tfrac{1}{2}kx^2\right) = 0 \quad \Rightarrow \boxed{h = \frac{kx^2}{2mg}}$$

Problem #6.6: Convert U_S to K

A spring with mass m attached is stretched a distance d. If the table is frictionless, then find the velocity of the mass when it reaches the relaxed position of the spring.

stretched position
$x_i = d$

relaxed position
$x_f = 0$



$$\Delta K + \Delta U + f\Delta s = W_{\text{ext}} \qquad \text{where } W_{\text{ext}} = f = 0$$

$$\left(K_f - K_i\right) + \left(U_{s,f} - U_{s,i}\right) = 0$$

$$\left(\tfrac{1}{2}mv_f^2 - 0\right) + \left(0 - \tfrac{1}{2}kd^2\right) = 0 \quad \Rightarrow \boxed{v_f = d\sqrt{\frac{k}{m}}}$$

What are units of $\sqrt{\frac{k}{m}}$?

Question #6.18: Convert K to U_S

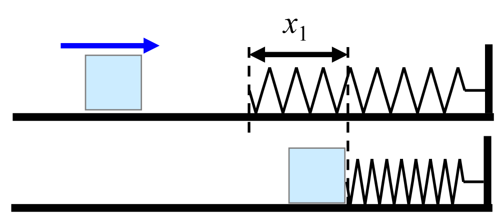
A box sliding on a horizontal frictionless surface runs into a fixed spring, compressing it a distance x_1 from its relaxed position and momentarily stopping. If the initial speed of the box were doubled and its mass were halved, how far x_2 would the spring now compress ?

(a) $x_2 = x_1$

(b) $x_2 = \frac{1}{2} x_1$

(c) $x_2 = 2x_1$

(d) $x_2 = \sqrt{2}\, x_1$

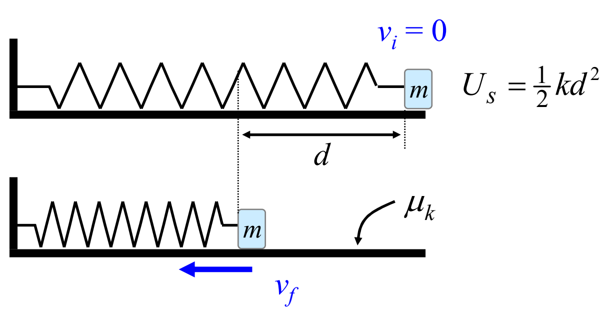


Problem #6.7: Convert U_S to K and Heat

A spring with mass m attached is stretched a distance d. If there is a coefficient of friction μ_k between the mass and table, then find the velocity of the mass when it reaches the relaxed position of the spring.

stretched position
$x_i = d$

relaxed position
$x_f = 0$



$$\Delta K + \Delta U + f\Delta s = W_{ext}$$

$$\left(\tfrac{1}{2}mv_f^2 - 0\right) + \left(0 - \tfrac{1}{2}kd^2\right) + \left(\mu_k mg\right)d = 0 \quad \Rightarrow \quad \boxed{v_f = \sqrt{\frac{kd^2}{m} - 2\mu gd}}$$